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1999 J. Phys. A: Math. Gen. 32 2949

(http://iopscience.iop.org/0305-4470/32/16/005)

J. Phys. A: Math. Gen. **32** (1999) 2949–2959. Printed in the UK PII: S0305-4470(99)97716-8

Griffiths singularities and the replica instantons in the random ferromagnet

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Received 21 September 1998

Abstract. The problem of existence of non-analytic (Griffiths-like) contributions to the free energy of a weakly disordered Ising ferromagnet is studied from the point of view of replica theory. The consideration is undertaken in terms of the usual random temperature Ginzburg–Landau Hamiltonian in space dimensions $D < 4$ in the zero external magnetic field. It is shown that in the paramagnetic phase, at temperatures not too close to T_c (where the behaviour of the pure system is correctly described by the Gaussian approximation), the free energy of the system has additional non-perturbative contributions of the form $\exp\{-(\mathrm{const})\tau^{(4-D)/2}/u\}$ (where $\tau = (T - T_c)/T_c$), which has an essential singularity in the parameter $u \to 0$ which describes the strength of the disorder. It is demonstrated that this contribution appears due to nonlinear localized (instanton-like) solutions of the mean-field stationary equations which are characterized by the special type of the replica symmetry breaking. It is argued that physically these replica instantons describe the contribution from rare spatial 'ferromagnetic islands' in which the local (random) temperature is below T_c.

1. Introduction

According to the original statement of Griffiths [1], the free energy of the random Ising ferromagnet in the temperature interval above its ferromagnetic phase transition point T_c and below the critical point $T_c^{(0)}$ of the corresponding pure system must be a non-analytic function of the external magnetic field h, such that in the limit $h \to 0$ the free energy as a function of h has essential singularity. Since this type of phenomenon, namely, the existence of non-analytic non-perturbative contributions to thermodynamical functions in random systems, seems to be rather a general one, at present it has become common to call any such contribution the 'Griffiths singularity'.

Due to intensive theoretical [2] and numerical [3] studies of the Griffiths singularities it was also discovered that the *dynamical* properties of the system in the temperature interval $T_c < T < T_c^{(0)}$ are not just ordinary paramagnetic. According to numerical simulations, the time correlation functions here can be described in terms of the so-called stretched-exponential asymptotic behaviour ($\sim \exp\{-(\mathrm{const})t^{\lambda(T)}\}$ with $\lambda < 1$), which is different from the usual exponential one, as it should be in the paramagnetic phase. On the other hand, recent analytical calculations [4] yield a time decay of the form $\exp\{-(\mathrm{const})(\ln t)^{D/(D-1)}\}$ (where D is the

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0305-4470/99/162949+11$19.50

dimensionality of the system). To underline that the properties of the system in the temperature interval $T_c < T < T_c^{(0)}$ are not quite paramagnetic, it has become common to call the state of the system here the 'Griffiths phase'.

At the level of 'hand-waving arguments' the dynamical Griffiths phenomena can be explained 'theoretically' rather easily: considering, for example, the bond diluted Ising model, one can note that at temperatures below $T_c^{(0)}$ in the 'ocean' of the zero magnetization paramagnetic background the random system must contain disconnected locally ordered 'ferromagnetic islands' (composed only of the pure system bonds) of all sizes, which, in turn, create the whole spectrum (up to infinity) of relaxation times. Having an infinite spectrum of relaxation times, with some imagination, it is not difficult to derive any relaxation law one likes, and the stretched-exponential one in particular.

Although it is commonly believed that the main point of the above 'explanation', namely the existence of an infinite number of local minima states, must be a general key for understanding the Griffiths phenomena (both dynamical and statistical mechanical), despite many efforts during the last 30 years, it has turned out to be extremely difficult to construct a more or less elaborated and convincing theory. For this reason any progress towards understanding the effects produced by numerous local minima states (which, so to say, are away from the perturbative region) looks valuable.

In this paper, we study non-perturbative contributions to the thermodynamical functions of a weakly disordered (random temperature) D-dimensional ($D < 4$) Ising ferromagnet in the paramagnetic phase away from the critical point. In the continuous limit this system can be described by the usual Ginzburg–Landau Hamiltonian:

$$H[\phi(x); \delta\tau(x)] = \int d^D x[\tfrac{1}{2}(\nabla\phi(x))^2 + \tfrac{1}{2}(\tau - \delta\tau(x))\phi^2(x) + \tfrac{1}{4}g\phi^4(x)]. \tag{1.1}$$

Here $\tau \equiv (T - T_c)/T_c \ll 1$ is the reduced temperature, and the quenched disorder is described by random spatial fluctuations of the local transition temperature $\delta\tau(x)$ whose probability distribution is taken to be symmetric and Gaussian:

$$P[\delta\tau] = p_0 \exp\left(-\frac{1}{4u}\int d^D x(\delta\tau(x))^2\right) \tag{1.2}$$

where $u \ll g$ is the small parameter which describes the strength of the disorder, and p_0 is an irrelevant normalization constant. For notational simplicity, we define the sign of $\delta\tau(x)$ in equation (1.1) so that positive fluctuations lead to locally ordered regions, whose effects will be the object of our further study.

As far as the corresponding pure system ($u \equiv 0$) is concerned, it is well known that in the close vicinity of T_c, at $|\tau| \ll \tau_g \sim g^{2/(4-D)}$, its properties are defined by non-Gaussian critical fluctuations (which can be studied, e.g., in terms of the ϵ-expansion renormalization group approach), while away from T_c, at $|\tau| \gg \tau_g$, the situation becomes Gaussian, and everything becomes very simple. Here the total magnetization of the system is defined by the order parameter $\langle\phi\rangle \equiv \phi_0(\tau)$ which is equal to zero above T_c, and is equal to $\pm\sqrt{|\tau|/g}$ below T_c; the asymptotic behaviour of the correlation function $G(x - x') \equiv (\langle\phi(x)\phi(x')\rangle - \phi_0^2)$ is defined only by the Gaussian fluctuations: $G(x) \sim |x|^{-(D-2)}$; and the singular part of the free energy $f(\tau)$ scales with the temperature as $f(\tau) \simeq \tau^{D/2}$.

Usually, the random system, defined by the Hamiltonian (1.1), is studied from the point of view of the effects produced by the quenched disorder on the critical phenomena in close vicinity to the phase transition point. A renormalization group consideration shows that if the temperature is not too close to T_c, at $\tau_u \ll \tau \ll \tau_g$ (where the disorder-dependent crossover temperature scale $\tau_u \sim u^{1/\alpha}$ is defined by the specific heat critical exponent $\alpha > 0$ of the pure system) the critical behaviour is essentially controlled by the pure system fixed point, and the disorder produces only irrelevant corrections. On the other hand, in close vicinity to the critical point, at $\tau \ll \tau_u$, the critical behaviour moves into a new universality class defined by the so-called random fixed point, which turns out to be universal [5]. In recent years, however, this very nice physical picture has been questioned on the grounds that the renormalization group approach completely misses the presence of numerous local minima configurations of the random Hamiltonian (1.1), which, in principle, may cause spontaneous replica symmetry breaking in the interaction parameters of the critical fluctuations, which, in turn, may ruin the above physical scenario [6].

Leaving the discussion of this very difficult problem for future analysis, in this paper we would like to pose a much simpler question: How do the thermodynamic functions of this system depend on the strength of the disorder u (in the limit $u \to 0$) far away from T_c, at $\tau \gg \tau_g$, where the behaviour of the pure system is correctly described by the Gaussian approximation? It turns out that even this, which seems an almost trivial question, is not so easy to answer.

Of course, first, one can proceed in a straightforward way, by developing the perturbation theory in powers of the parameter u at the background of the pure system paramagnetic state $\langle\phi\rangle = 0$ using the Gaussian approximation for the thermal fluctuations. There is nothing wrong in this approach, but the problem is that it cannot give *all* the thermodynamic contributions which exist at $u \neq 0$. The drawback with this type of perturbation theory is the same as that of the renormalization group: it completely misses the existence of numerous (a macroscopic number) local minima configurations of the random Hamiltonian (1.1).

It is worth noting that even in the case of the zero-dimensional version of the system considered here, the free energy as a function of u was shown to have essential singularity in the limit $u \to 0$. The explicit form of this singularity has been calculated analytically, first in terms of the vector replica symmetry breaking ansatz by Bray *et al* [7], and subsequently this result has been confirmed without the use of replicas by McKane [8].

At the level of 'hand waving arguments' it is very easy to see what all these off-perturbative states are. At any $u \neq 0$ there exists a finite (exponentially small) density of 'ferromagnetic islands' in which the local (random) temperature is below T_c (such that $\delta\tau(x) > \tau$), and the minimum energy configurations here are achieved at a non-zero local value of the order parameter: $\phi_0(x) \sim \pm\sqrt{(\delta\tau - \tau)/g}$. Since the spatial density of such islands is finite and each island provides two ($\pm$) possibilities for the local magnetization, the total number of local minima configurations in the system must be exponential in its volume.

Formally, to take into account the contributions of all these states, one has to proceed as follows. For an arbitrary quenched function $\delta\tau(x)$ one has to find all possible local minima solutions of the saddle-point equation:

$$-\Delta\phi(x) + (\tau - \delta\tau(x))\phi(x) + \phi(x)^3 = 0. \tag{1.3}$$

Then one has to substitute these solutions into the Hamiltonian (1.1) and calculate the corresponding thermodynamic weights. Next, to compute the partition function one has to perform summation over all the solutions, and finally to get the corresponding free energy one has to take the logarithm of the partition function and average it over random functions $\delta\tau(x)$ with the probability distribution (1.2). Clearly, it is hardly possible that such a program can be implemented.

On the other hand, as usual, for the systems which contain quenched disorder we can use the standard replica method and reduce the problem of the quenched averaging to the annealed one for n copies of the original system:

$$F = -\overline{(\ln Z)} = -\lim_{n\to 0}\frac{1}{n}(\overline{Z^n} - 1) \tag{1.4}$$

where $\overline{(\ldots)}$ denotes the averaging over random functions $\delta\tau(x)$ with the probability distribution (1.2), and

$$Z[\delta\tau(x)] = \int \mathcal{D}\phi(x) \exp(-H[\phi(x); \tau(x)]) \tag{1.5}$$

is the disorder-dependent partition function which is given by the functional integration over configurations of the field $\phi(x)$.

Simple Gaussian integration over $\delta\tau(x)$ in equation (1.4) yields

$$\overline{Z^n} = \prod_{a=1}^{n} \left[\int \mathcal{D}\phi_a(x) \right] \exp(-H_n[\phi_a(x)]) \tag{1.6}$$

where

$$H_n[\phi_a(x)] = \int \mathrm{d}^D x \left[\frac{1}{2} \sum_{a=1}^{n} (\nabla\phi_a)^2 + \frac{1}{2}\tau \sum_{a=1}^{n} \phi_a^2 + \frac{1}{4} g \sum_{a=1}^{n} \phi_a^4 - \frac{1}{4} u \sum_{a,b=1}^{n} \phi_a^2 \phi_b^2 \right] \tag{1.7}$$

is the *spatially homogeneous* replica Hamiltonian.

Now, if we are intended to take into account non-trivial local minima states, instead of solving the original inhomogeneous stationary equation (1.3), we can consider the corresponding replica saddle-point equations:

$$-\Delta\phi_a(x) + \tau\phi_a(x) + \phi_a^3(x) - u\phi_a(x) \sum_{b=1}^{n} \phi_b^2(x) = 0. \tag{1.8}$$

Since until now all the transformations have been exact, these equations must contain (maybe in a slightly hidden way) all the relevant non-trivial states which in the language of the original random Hamiltonian correspond to rare ferromagnetic islands.

At this stage we can note one very simple point. Looking for various types of solutions of the above equations one can first try the simplest possible 'replica symmetric' ansatz, in which the fields in all replicas are assumed to be equal: $\phi_a(x) = \phi(x)$. In this case the last term in equations (1.8) (which contains the factor $\sum_{b=1}^{n} \phi_b^2(x) = n\phi^2(x)$) drops away in the limit $n \to 0$, and these equations reduce to the *pure system* saddle-point equation

$$-\Delta\phi(x) + \tau\phi(x) + \phi(x)^3 = 0 \tag{1.9}$$

which at $\tau > 0$ has only the trivial solution $\phi(x) \equiv 0$. This means that in any non-trivial solution of equations (1.8) the fields $\phi_a(x)$ in different replicas *cannot* all be equal. In other words, the symmetry among replicas in the *replica vector* $\phi_a(x)$ must be broken.

The methodological aspects of how to handle the vector replica symmetry breaking situation in various disordered systems are described in [9]. In the next section this method will be applied to the problem described above. It will be shown that, indeed, in the high-temperature region ($\tau > 0$) equations (1.8) have non-trivial localized (having finite size and finite energy) solutions in which the replica symmetry in the fields $\phi_a(x)$ is broken. The formal summation over all such solutions provides the contribution to the free energy of the typical Griffiths-like form: $\exp\{-(\mathrm{const})\tau^{(4-D)/2}/u\}$. It will also be shown that the mean-field approach (in which the critical fluctuations are ignored) used in this paper is grounded only if the temperature is not too close to T_c, namely at $\tau \gg \tau_g \sim g^{2/(4-D)}$, the same as in classical Ginzburg–Landau theory. Finally, it will be demonstrated how this type of non-analytic contribution to the free energy can be estimated from purely physical arguments taking into account probabilities for the typical 'ferromagnetic islands'.

To avoid possible misunderstandings, as a conclusion to this introductory section we would like to note the following essential point. The problem considered in this paper is actually rather far from the original one studied by Griffiths as well as by many other people later on.

Since the shift of T_c in the weakly disordered ferromagnet compared to $T_c^{(0)}$ of the pure system is of the order of $\sqrt{u}$, in the limit $u \ll g$ the interval of temperatures $T_c < T < T_c^{(0)}$, where the so-called Griffiths phase is expected to take place, appears to be well inside the temperature interval $\tau_g \sim g^{2/(4-D)}$ where the critical fluctuations are essential, and where the mean-field approach considered in this paper cannot be used. For that reason, in the considered range of temperatures $\tau \gg \tau_g$ it is hardly reasonable to look for non-analytic behaviour of the free energy as a function of the external magnetic field (at least the present approach in terms of the replica instantons modified by the external field h does not seem to indicate on any non-analyticity in h). The aim of this paper is just to demonstrate that in addition to the 'usual' Griffiths singularities in terms of the external field, the free energy of the random ferromagnet (in the zero magnetic field) must also be non-analytic in the value of the parameter which describes the strength of the disorder.

2. Replica instantons

Following the general strategy developed in [9], let us assume that in addition to the trivial replica symmetric (RS) solutions of the saddle-point equations (1.8) there exist other types of solutions, which are *well separated* in the configurational space from the RS state. In this case, denoting the contribution of these non-trivial states by the label 'replica symmetry breaking' (RSB), the replica partition function, equation (1.6), can be decomposed into two parts

$$\overline{Z^n} = Z_{\mathrm{RS}} + Z_{\mathrm{RSB}} \tag{2.1}$$

where Z_{RS} contains all the perturbative contributions in the vicinity of the trivial state $\phi_a(x) = 0$. As usual, this partition function can eventually be represented in the form

$$Z_{\mathrm{RS}} = \exp(-nVf_{\mathrm{RS}}) \tag{2.2}$$

where V is the volume of the system and f_{RS} is the free energy density, which contains the pure system leading term $\sim\tau^{D/2}$ (at temperatures not too close to T_c, $\tau \gg \tau_g$), plus the perturbation series in powers of the disorder parameter u.

Thus, in terms of the general replica approach, according to equation (1.4) for the total free energy we get

$$F = Vf_{\mathrm{RS}} + F_{\mathrm{RSB}} \tag{2.3}$$

where the additional RSB part of the free energy

$$F_{\mathrm{RSB}} = -\lim_{n\to 0}\frac{1}{n}Z_{\mathrm{RSB}} \tag{2.4}$$

must contain all non-perturbative contributions (if any) which are away from the trivial state $\phi_a = 0$. It is this part of the free energy which will be a point of our further study.

The simplest possible non-trivial replica structure for the solutions of the saddle-point equations (1.8) can be taken in the following form (see [9])

$$\phi_a(x) = \begin{cases} \pm\phi(x) & \text{for } a = 1, \ldots, k \\ 0 & \text{for } a = k+1, \ldots, n \end{cases} \tag{2.5}$$

where k is the integer value parameter: $k = 1, 2, \ldots, n$ which defines a given structure of the trial replica vector ϕ_a (note that the value $k = 0$ should be excluded since it describes the trivial RS solution which is already taken into account in f_{RS}). The solutions in equation (2.5) are taken with the '$\pm$' signs, since the saddle-point equations (1.8) are invariant with respect to the global change of signs of the replica fields.

Substituting this ansatz into equations (1.8) as well as into the replica Hamiltonian (1.7), one finds that for a given value of the parameter k the fields $\phi(x)$ in equation (2.5) are defined by the solutions of the following saddle-point equation

$$-\Delta\phi(x) + \tau\phi(x) - \lambda(k)\phi(x)^3 = 0 \tag{2.6}$$

and the thermodynamic weight of any such solution is defined by the energy:

$$E(k) = k\int \mathrm{d}^D x[\tfrac{1}{2}(\nabla\phi(x))^2 + \tfrac{1}{2}\tau\phi^2(x) - \tfrac{1}{4}\lambda(k)\phi^4(x)] \tag{2.7}$$

where

$$\lambda(k) = (uk - g). \tag{2.8}$$

Summing over the parameter k and taking into account the combinatorial factor, which is the number of permutations among replicas in the ansatz structure (2.5) for the free energy, equation (2.4), one obtains

$$F_{\mathrm{RSB}} = -\lim_{n\to 0}\frac{1}{n}\sum_{k=1}^{n}\frac{n!}{k!(n-k)!}2^k\exp\{-E(k)\} \tag{2.9}$$

(the factor 2^k appears due to independent summation over $\pm$ signs, in k non-zero replicas, equation (2.5)). To take the limit $n \to 0$ the series in the above equation can be represented as follows:

$$F_{\mathrm{RSB}} = -\lim_{n\to 0}\frac{1}{n}\sum_{k=1}^{\infty}\frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}2^k\exp\{-E(k)\}. \tag{2.10}$$

Here the summation with respect to k is extended beyond $k = n$ to ∞ since the gamma function is equal to infinity at negative integers. Now using the relation $\Gamma(-z) = \pi[z\Gamma(z)\sin(\pi z)]^{-1}$, we can perform the analytic continuation $n \to 0$:

$$\begin{aligned}\frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} &= \left.\frac{\Gamma(n+1)(k-1-n)\Gamma(k-1-n)\sin(\pi(k-1-n))}{\pi\Gamma(k+1)}\right|_{(n\to 0)}\\ &\simeq n\frac{(-1)^{k-1}}{k}.\end{aligned} \tag{2.11}$$

Thus, for the free energy (2.9) one obtains

$$F_{\mathrm{RSB}} = -\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}2^k\exp\{-E(k)\}. \tag{2.12}$$

At this stage we can note the following important point. For any *non-localized* (e.g. space-independent) solution, such that its energy (2.7) is divergent with the volume V of the system, the corresponding contribution to the free energy (2.12) will not be proportional to V, but instead it will contain the volume in the exponential factor. This means that at least for the bulk properties of the system this type of solution must be irrelevant.

Thus, we have to look for *localized* solutions: those which are local in space (breaking translation invariance) and which have *finite* (volume-independent) energy. Let us suppose that such instanton-type solutions do exist (see later), and that for a given k the solution is characterized by the spatial size $R(k)$. Then, if we take into account only one-instanton contribution (or in other words if we consider a gas of *non-interacting* instantons), due to the obvious entropy factor V/R^D (which is the number of positions of the object of the size R in the volume V) we get the free energy proportional to the volume:

$$F_{\mathrm{RSB}} \simeq -V\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}R^{-D}(k)2^k\exp\{-E(k)\}. \tag{2.13}$$

Now let us come back to the saddle-point equation (2.6), and let us consider the range of the parameter k such that $\lambda(k) = (uk - g) > 0$ (i.e. $k > [g/u]$). Rescaling the fields

$$\phi(x) = \sqrt{\frac{\tau}{\lambda(k)}}\psi(x\sqrt{\tau}) \tag{2.14}$$

instead of equation (2.6) one obtains the following differential equation which contains no parameters:

$$-\Delta\psi(z) + \psi(z) - \psi^3(z) = 0. \tag{2.15}$$

Correspondingly, for the energy, equation (2.7), one obtains

$$E(k) = \frac{k}{uk - g}\tau^{(4-D)/2}E_0 \tag{2.16}$$

where

$$E_0 = \int \mathrm{d}^D z[\tfrac{1}{2}(\nabla\psi(z))^2 + \tfrac{1}{2}\psi^2(z) - \tfrac{1}{4}\psi^4(z)]. \tag{2.17}$$

Equation (2.15) is well known in field theory (see, e.g., [10]): for the present choice of signs of the linear and the cubic terms (which imposes the conditions $\tau > 0$ and $k > [g/u]$) in dimensions $D < 4$ this equation has spherically symmetric instanton-like solutions such that

$$\begin{aligned}&\psi(|z| \leqslant 1) \simeq \psi(0) \sim 1\\ &\psi(|z| \gg 1) \sim \exp(-|z|) \to 0.\end{aligned} \tag{2.18}$$

The energy, equation (2.17), of such a solution is a finite and *positive* number. Of course, for a *generic* value of the field $\psi(0)$ at the origin, the solution tends to the values $\psi(|z| \to \infty) = \pm 1$ which are the extrema of the potential $\frac{1}{2}\psi^2 - \frac{1}{4}\psi^4$, and any such solution has divergent energy (2.17). However, there exists a discrete set of initial values ψ_0 for which the solution (exponentially) tends to zero at infinity, and which has finite energies. It can be shown that the solution with minimal energy E_0 corresponds to the minimal value of $|\psi_0|$ in the set. In particular, at $D = 3$, $\psi_0 \simeq 4.34$ and $E_0 \simeq 18.90$. For our further calculations with exponential accuracy it will be sufficient to take into account only the solution with the minimal energy.

According to the rescaling (2.14), in terms of the original fields $\phi(x)$ the size of the instanton is $R = \tau^{-1/2}$ (note that it does not depend on k), which coincides with the usual correlation length of Ginzburg–Landau theory. Substituting this value of R as well as the energy (2.16) of the instanton into the series (2.13) for the free energy one gets

$$F_{\mathrm{RSB}} \simeq -V\tau^{D/2}\sum_{k>[g/u]}^{\infty}\frac{(-1)^{k-1}}{k}2^k\exp\left[-E_0\frac{k}{uk-g}\tau^{(4-D)/2}\right]. \tag{2.19}$$

It can easily be shown that under the considered conditions on the parameters u, g and τ ($u \ll g \ll 1$, and $g^{2/(4-D)} \ll \tau \ll 1$) the leading contribution in the above series with exponential accuracy comes from the region $k \gg g/u \gg 1$:

$$\frac{1}{V}F_{\mathrm{RSB}} \simeq \tau^{D/2}\exp\left[-E_0\frac{\tau^{(4-D)/2}}{u}\right]\sum_{k\gg g/u}^{\infty}\frac{(-1)^{k-1}}{k}2^k. \tag{2.20}$$

Here the absolute value of the series $\sum_{k=k_0\gg 1}^{\infty}k^{-1}(-1)^{k-1}2^k$ can be estimated by the upper bound $\sim k_0^{-1}2^{k_0}$, and since it is assumed that $\tau \gg g^{2/(4-D)}$ the term $(g/u)\ln 2$, which appears in the exponential, can be dropped in comparison with $E_0\tau^{(4-D)/2}/u$. Thus, for the density of the free energy we finally obtain the following contribution

$$\frac{1}{V}F_{\mathrm{RSB}} \sim \exp\left[-E_0\frac{\tau^{(4-D)/2}}{u}\right] \tag{2.21}$$

(where we have dropped all pre-exponential factors, which within the present accuracy of calculations cannot be defined).

3. Fluctuations

Note, first, that one should not be confused by the 'wrong' sign of the ϕ^4 interaction term in the energy function (2.7), which for the usual field theory would indicate its absolute instability. Here, as is usual in replica theory, in the limit $n \to 0$ everything turns 'up down', so that the minima of the physical free energy actually correspond to the *maxima* of the replica free energy. It can be easily shown (see later) that formal integration over n-component replica fluctuations around the considered instanton solution in the limit $n \to 0$ yields a physically sensible result.

Proceeding the same way as in usual Ginzburg–Landau theory, let us determine under which conditions the above mean-field approach used to derive the result (2.21) can be valid. Introducing small fluctuations $\varphi_a(x)$ near the instanton solution, equations (2.5) and (2.14), $\phi_a(x) = \phi_a^{(\mathrm{inst})}(x) + \phi_a(x)$, in the Gaussian approximation we get the following Hamiltonian for the fluctuating fields

$$H[\varphi] = \int \mathrm{d}^D x \left[\frac{1}{2}\sum_{a=1}^{n}({}_a(x))^2 + \frac{1}{2}\tau \sum_{a,b=1}^{n} T_{ab}(x)\varphi_a(x)\varphi_b(x)\right] \tag{3.1}$$

where the matrix $T_{ab}(x)$ contains the $k \times k$ block

$$T_{ab}^{(k)}(x) = \left(1 - \frac{uk-3g}{uk-g}\psi^2(x\sqrt{\tau})\right)\delta_{ab} - \frac{2u}{uk-g}\psi^2(x\sqrt{\tau}) \tag{3.2}$$

(where $a, b = 1, \ldots, k$) and the diagonal elements for the remaining $(n-k)$ replicas

$$T_{ab}^{(n-k)} = \left(1 - \frac{uk}{uk-g}\psi^2(x\sqrt{\tau})\right)\delta_{ab} \tag{3.3}$$

(where $a, b = k+1, \ldots, n$). Here the function $\psi(z)$ is the instanton solution, equation (2.18).

Since the mass term in the Hamiltonian (3.1) is proportional to τ, the behaviour of the correlation function of the fluctuating fields at scales $|x| \ll R_{\mathrm{c}} \sim \tau^{-1/2}$ appears to be the same as in Ginzburg–Landau theory: $G_{ab}(x-x') = \langle\varphi_a(x)\varphi_b(x')\rangle \sim |x-x'|^{-(D-2)}\delta_{ab}$ (beyond R_{c} this correlation function decays exponentially). Therefore, the typical value of the fluctuations $\langle\varphi^2\rangle$ can be estimated in the usual way:

$$\langle\varphi^2\rangle \sim \frac{1}{n}\sum_{a=1}^{n} R_{\mathrm{c}}^{-D}\int_{|x|<R_{\mathrm{c}}} \mathrm{d}^D x\, G_{aa}(x) \sim \tau^{(D-2)/2}. \tag{3.4}$$

The saddle-point approximation considered in the previous section is justified only if the typical value of the fluctuations is small compared to the value of the 'background' instanton field $\phi^{(\mathrm{inst})}(x) \sim \sqrt{\tau/\lambda(k)}$ (see equation (2.14)):

$$\tau^{(D-2)/2} \ll \frac{\tau}{\lambda(k)} \Rightarrow \lambda(k) \sim uk \ll \tau^{(4-D)/2}. \tag{3.5}$$

On the other hand, the contribution (2.21) appears due to summation in the region $k \gg g/u$. Thus, one can get this type of contribution to the free energy only in the following interval of summation with respect to k:

$$\frac{g}{u} \ll k \ll \frac{1}{u}\tau^{(4-D)/2}. \tag{3.6}$$

This interval exists provided

$$\tau \gg g^{2/(4-D)} \tag{3.7}$$

which is the usual Ginzburg–Landau criteria.

One can also arrive at the same conclusion by deriving the fluctuational contribution to the RSB part of the free energy by direct integration over the fluctuating fields using the Gaussian Hamiltonian (3.1) (this way one can also check that this contribution contains no imaginary parts which would happen, if the considered extrema would correspond to a physically unstable field configuration). Assuming the θ-like structure of the instanton solution, $\psi(|z| \leqslant 1) \simeq \psi(0) \equiv \psi_0 \sim 1$ and $\psi(|z| > 1) = 0$, the fluctuating modes with momenta $p \ll \sqrt{\tau}$ and $p \gg \sqrt{\tau}$ in the Hamiltonian (3.1) can be explicitly decoupled

$$H = \frac{1}{2}\sum_{a,b=1}^{n}\int_{|p|\gg\sqrt{\tau}} \frac{\mathrm{d}^D p}{(2\pi)^D}[p^2\delta_{ab} + \tau T_{ab}]\varphi_a(p)\varphi_b(-p) + \frac{1}{2}\sum_{a=1}^{n}\int_{|p|\ll\sqrt{\tau}} \frac{\mathrm{d}^D p}{(2\pi)^D} p^2|\varphi_a(p)|^2 \tag{3.8}$$

where the p-independent matrix T_{ab} is given by equations (3.2) and (3.3), in which instead of the function $\psi(x\sqrt{\tau})$ one has to substitute the constant ψ_0.

The integration over the *replica symmetric* modes with momenta $p \ll \sqrt{\tau}$ (they correspond to fluctuations at scales much bigger than the size of the instanton), described by the second term of the Hamiltonian (3.8), gives the contribution of the form $\exp(-nV\tilde{f}_{\mathrm{RS}})$, and it vanishes in the limit $n \to 0$ (note that in the RSB part of the free energy we have to keep only the terms which remain *finite* in the limit $n \to 0$ and not linear in n). This is natural, because this contribution is already contained in the RS part of the free energy.

The integration over the modes with momenta $p \gg \sqrt{\tau}$ is slightly cumbersome but straightforward:

$$\begin{aligned}\tilde{Z}_{\mathrm{RSB}} &\equiv \prod_{p\gg\sqrt{\tau}}\left[\int \mathcal{D}\varphi_a(p)\right]\exp\{-H[\varphi_a(p)]\} \\ &= \exp\left[-\frac{1}{2}\tau^{-D/2}\int_{p\gg\sqrt{\tau}} \mathrm{d}^D p\,\mathrm{Tr}\ln(p^2\delta_{ab} + \tau T_{ab})\right].\end{aligned} \tag{3.9}$$

The matrix under the logarithm in the above equation contains $(k-1)$ eigenvalues

$$\lambda_1 = p^2 + \tau\left(1 - \frac{uk-3g}{uk-g}\psi_0^2\right) \tag{3.10}$$

one eigenvalue

$$\lambda_2 = p^2 + \tau\left(1 - \frac{uk-3g}{uk-g}\psi_0^2\right) - \tau\frac{2uk}{uk-g}\psi_0^2 \tag{3.11}$$

and $(n-k)$ eigenvalues

$$\lambda_3 = p^2 + \tau\left(1 - \frac{uk}{uk-g}\psi_0^2\right). \tag{3.12}$$

Substituting these eigenvalues into equation (3.9), after simple algebra in the limit $n \to 0$ one eventually obtains the following result:

$$\tilde{Z}_{\mathrm{RSB}} \sim \exp\left(\frac{3k}{2(uk-g)}g\psi_0^2\right). \tag{3.13}$$

Thus we see that in the region $\tau \gg g^{2/(4-D)}$ the factor $kg/(uk-g)$ in the exponential of the above equation is small compared to the leading term $k\tau^{(4-D)/2}/(uk-g)$ given by the saddle-point solution, equation (2.16).

4. Discussion

It is interesting to note that the non-analytic instanton contribution of the form given by equation (2.21) can be easily 'derived' based on qualitative physical arguments. Let us again consider the random Hamiltonian (1.1) at temperatures above T_c ($\tau > 0$), and let us estimate the contribution to the free energy coming from rare 'ferromagnetic islands' where $\delta\tau(x) > \tau$. In the mean-field regime at finite values of τ the typical smallest (most probable) size of such an island is $R_c \sim \tau^{-1/2}$. Therefore, according to the probability distribution, equation (1.2), in the limit of weak disorder ($u \to 0$) the contribution of the islands to the free energy with exponential accuracy can be estimated by their probability

$$\begin{aligned}\delta F &\sim \int_\tau^\infty \mathrm{d}(\delta\tau) \exp\left(-\frac{(\mathrm{const})}{u}\tau^{-D/2}(\delta\tau)^2\right) \\ &\sim \exp\left(-\frac{(\mathrm{const})}{u}\tau^{(4-D)/2}\right) \end{aligned} \tag{4.1}$$

which (up to the undefined (const) factor) coincides with the result (2.21).

The above qualitative consideration seems rather valuable because it provides good physical support for a more exact but slightly formal and somewhat mysterious vector replica symmetry breaking scheme considered in section 2.

Of course, exponentially small contributions to the free energy (as well as to other thermodynamical functions) of the type (2.21) are not so important for thermodynamical properties of the random ferromagnet in the considered paramagnetic temperature region. Nevertheless, the fact of their existence seems very interesting for two reasons.

First, it tells us that even in the mean-field regime the free energy of the random ferromagnet must be a non-analytic function of the parameter which describes the strength of disorder $u \to 0$, which is interesting in itself.

Second, it indicates the importance of nonlinear excitations which in terms of the present replica field theoretical approach are described by the localized instanton-like solutions of the stationary equations. In the considered mean-field region away from T_c these excitations provide only exponentially small corrections. However, in the close vicinity of the critical point the presence of instantons (which is ignored in the standard renormalization-group approach), and their interactions with the critical fluctuations may produce a dramatic effect on the critical properties of the phase transition. It is worth noting that although in the scaling regime (at $T = T_c$) the situation looks very different from that considered in this paper, the corresponding stationary equations (1.8) (with $\tau = 0$) also have nonlinear instanton-like solutions with the RSB structure given by equation (2.5). One can easily check that in the dimension $D = 4$ these solutions can be found explicitly [11]

$$\phi(x) = \sqrt{\frac{8}{(uk-g)}}\frac{R}{R^2 + |x|^2} \tag{4.2}$$

where the size of the instanton R appears to be the *zero mode* (the energy of the instanton does not depend on R). In dimensions below but close to four (at $\epsilon = (4-D) \ll 1$) the field configuration given by equation (4.2) can be considered as the approximate solution which contains the parameter R as the *soft mode*, since the energy of the instanton, equation (2.7), depends on R very weakly

$$E(k) = \frac{4}{3}S_D R^{-\epsilon}\frac{k}{uk-g} \tag{4.3}$$

(here S_D is the square of the unit D-dimensional sphere).

At present it is not quite clear how all these nonlinear instanton excitations could be incorporated into the self-consistent theory of the critical fluctuations. Keeping in mind the fact that the degrees of freedom of this type explicitly break the replica symmetry, a kind of 'heuristic' renormalization group approach has been proposed [6], in which it was assumed that due to interactions of the fluctuations with this type of non-perturbative excitations the replica symmetry in the effective matrix, describing nonlinear interactions of the fluctuating fields, is spontaneously broken. This resulted in the instability of previously known fixed points and remarkable 'runaway' behaviour of the renormalization group flows (which, e.g., may indicate the onset of a kind of glass-like phase in a narrow temperature interval around T_c). We hope that the study described in the present paper stimulates further much deeper investigation of the physics of the phase transition in random ferromagnets.

Acknowledgments

The author is grateful to M Mézard, Vl Dotsenko, G Parisi and S Franz for useful discussions. This work has been supported in part by the Russian Fund for Fundamental Research (grant Nos 96-15-96920 and 96-02-018985).

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